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THE PLANE PROBLEM OF THE EXTRUSION OF A VISCOPLASTIC MEDIUM BY PARALLEL PLATES†

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A solution of the problem of the plane parallel flow of viscoplastic medium between two parallel plates when they approach (separate) at a specified velocity is given within the framework of the Bingham model in the inertialess thin-layer approximation for arbitrary values of the coefficient of viscosity and the yield stress. Analytic expressions are obtained for the velocity and pressure fields. The boundary of the flow kernel, where the shear stress on the areas of the parallel planes of the plates is less than the yield stress and the component of the velocity, parallel to the plates, does not change in a transverse direction, is determined. A single similarity parameter which defines the kinematic and dynamic flow characteristics is found. For a specified law of motion of the plates, a general expression is obtained for the force acting on plates of finite size in terms of a dimensionless function of a single dimensionless parameter. The law of approach (separation) of the plates under a constant force is found. © 1998 Elsevier Science Ltd. All rights reserved.

A solution of the problem of the extrusion of a viscous medium between approaching parallel plates was obtained in the thin-layer approximation of the hydrodynamic theory of lubrication in [1]. Prandtl constructed an exact solution in the case of the extrusion of a purely plastic medium [2]. A small correction to the Prandtl solution when the viscous stresses are small compared with the plastic stresses was obtained in [3]. This solution differs from the Prandtl solution in that there is a thin boundary layer close to the wall in which the deformations are non-zero.

A boundary-value problem on the extrusion of a viscoplastic medium between two parallel plates in the thin-player approximation (for a small ratio of the distance between the plates to the length of the plates) is formulated below, its variational formulation is given and an existence and uniqueness theorem is proved.

A solution of the boundary-value problem, which holds for any relations between the plasticity and viscosity, is constructed in this approximation. It is based on the exact solution of the problem of the viscoplastic flow between two fixed plates under a pressure gradient [4]. The result generalizes the solutions for a viscous medium [1], a purely plastic medium [2] and a viscoplastic medium with a low coefficient of viscosity [3] in the case of zero, infinitely high and high Saint-Venant numbers.

1. FLOW BETWEEN TWO FIXED PLATES

In the case of a viscoplastic medium, the relation between the stress tensor p_{ij} and the strain-tensor rate e_{ij} has the form [5]

$$p_{ij} = -p\delta_{ij} + \tau_{ij}; \quad \tau_{ij} = 2(\mu + \tau_0 / H)e_{ij} \tag{1.1}$$

$$H = (2e_{ij}e_{ij})^{1/2} > 0; \quad i, j = 1,2,3$$

where p is the pressure, μ is the coefficient of dynamic viscosity, τ_0 is the yield stress, and summation is over repeated subscripts i and j.

A simple exact solution of the problem of the flow of an incompressible viscoplastic medium is a plane flow between two fixed parallel plates under a pressure gradient is known [4], and it can be represented as follows (Fig. 1a).

The stress deviator τ_{ij} has just two components $\tau_{13} = \tau_{31} = \tau_{xz}$, the shear stress for the areas of the parallel plates, which is non-zero. The stream function ψ depends on the dimensionless distance $Z = z/h$ from one of the plates ($2h$ is the separation between the plates and Q is the flow rate)

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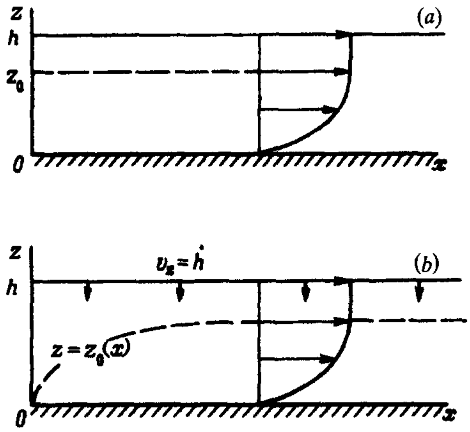

Fig. 1.

$$\psi = \frac{h^2\tau_0 \operatorname{sign} Q}{6\mu}\Psi(Z, Z_0) \tag{1.2}$$

$$\Psi = \frac{1}{1-Z_0}\times\begin{cases} Z^2(3Z_0 - Z), & 0 \leqslant Z \leqslant Z_0 \\ Z_0^2(3Z - Z_0), & Z_0 \leqslant Z \leqslant 1 \end{cases}$$

The velocity components are expressed in terms of the stream function

$$v_x = \partial\psi / \partial z; \quad v_z = -\partial\psi / \partial x = 0 \tag{1.3}$$

When $Z > 1$, the velocity is extended using symmetry.

It follows from (1.2) that the parabolic velocity profile when $Z \in [0, Z_0]$ is smoothly joined to the rectilinear profile when $Z \in [Z_0, 1]$. The boundary $Z = Z_0$ separates the domain of viscous flow from the domain of translational motion of the medium without deformation. The quantity Z_0, the pressure gradient and the shear stress are connected by the relations

$$\frac{\partial p}{\partial x} = -\frac{\operatorname{sign} Q\tau_0}{h(1-Z_0)}, \quad \tau_{xz} = (z-h)\frac{\partial p}{\partial x} \tag{1.4}$$

The flow rate $Q = 2(\psi(1) - \psi(0))$ is found from (1.2) and is expressed by the relation

$$a = \frac{\mu|Q|}{\tau_0 h^2} = \frac{Z_0^2(1 - \frac{1}{3}Z_0)}{1 - Z_0} \tag{1.5}$$

Hence, the quantity Z_0 is determined using formula (1.4) for a specified pressure gradient, and the velocity field and flow rate are determined using formulae (1.2) and (1.5).

The solution can be given another treatment which will also be used later to solve the problem of the extrusion of a viscoplastic medium. Suppose the flow rate Q is known and it is necessary to determine the pressure gradient to be applied in this case in order to find the velocity field and the quantity Z_0. In order to solve this problem, it is necessary to find the function $Z_0(a)$ from Eq. (1.5) and then the pressure gradient, using formula (1.4), and the velocity field, using (1.1). Equation (1.5) is a cubic equation in Z_0 and its single root in the interval $Z_0 \in [0, 1)$ can be represented by the series

$$Z_0 = 1 - 3\sum_{m=0}^{\infty} \frac{(m+1)b_m}{(3m+1)(a+1)^{3m+1}} \tag{1.6}$$

$$\frac{1}{1-Z_0} = \frac{3}{2}(a+1) - \sum_{m=0}^{\infty} \frac{b_m}{(a+1)^{3m+2}}$$

$$b_0 = \frac{2}{9}, \quad b_1 = \frac{2^4}{3^5}, \ldots, \quad b_m = \frac{2^{2m+1}(3m+1)!}{3^{3m+2}(2m+1)!(m+1)!}$$

Series (1.6) converge for all $a \geqslant 0$. In order to calculate the function Z_0 and $1 - Z_0$ when $0 \leqslant a \leqslant a_0$, it is more convenient to use the expansions

$$Z_0 = a^{1/2} - \frac{1}{3}a - \frac{1}{18}a^{3/2} + \frac{1}{27}a^2 + \ldots$$
$$\frac{1}{1-Z_0} = 1 + a^{1/2} + \frac{2}{3}a + \frac{5}{18}a^{3/2} + \frac{1}{27}a^2 + \ldots \tag{1.7}$$

(The values of the parameter a_0 and the corresponding best approximations are presented in Section 7.)

It will be shown below how these results can be extended to solve the problem of the extrusion of a viscoplastic medium between approaching parallel plates.

2. FORMULATION OF THE GENERAL PROBLEM. EXISTENCE AND UNIQUENESS THEOREMS

We will now consider the problem of the flow of an incompressible viscoplastic medium between two parallel plates which are approaching (separating) in a direction perpendicular to the plane of the plates (Fig. 1b).

When the dimensions of the plates l are very much greater than the distance $2h$ between them, we use the thin-layer approximation for a viscous fluid [1, 6]. The equations in the stresses can also be extended to a viscoplastic medium

$$\partial\tau_{xz} / \partial z = \partial p / \partial x, \quad \partial p / \partial z = 0 \tag{2.1}$$

We add to (2.1) the solution of the continuity equation, which expresses the velocity components in terms of the stream function (formula (1.3)) and the rheological relation (1.1). In the thin layer, account has to be taken of the fact that $\partial/\partial x \ll \partial/\partial z$ and that all of the components of the stress tensor, apart from the components $\tau_{13} = \tau_{31} = \tau_{xz}$, can be neglected. Equation (1.1) then takes the form

$$\tau_{xz} = \mu\partial^2\psi / \partial z^2 + \tau_0 \operatorname{sign}(\partial^2\psi / \partial z^2), \quad |\tau_{xz}| > \tau_0$$
$$\partial^2\psi / \partial z^2 = 0, \quad |\tau_{zx}| \leqslant \tau_0 \tag{2.2}$$

Finally, in a system of coordinates fixed to one of the plates, the following boundary conditions for the stream function follow from the no-slip conditions on the plates $z = 0$ and $z = 2h$

$$z = 0: \ \partial\psi/\partial z = 0, \quad \partial\psi/\partial x = 0$$
$$z = 2h: \ \partial\psi/\partial z = 0, \quad -\partial\psi/\partial x = 2\dot{h} \tag{2.3}$$

where $2h$ is the rate of change of the distance between the plates.

The origin of the system of coordinates $x = 0$ is placed on the axis of symmetry of the flow and it then follows that the conditions for the stream function and the pressure

$$\psi(0, z) = 0, \quad p(l, z) = p_l \tag{2.4}$$

should be imposed at the ends of the interval $(0, l)$.

The functions τ_{xz} and υ_x must be continuously differentiable with respect to z and, consequently, the stream function must have two continuous derivatives with respect to z.

Problem (2.1)–(2.4) has a unique solution for the stream function and consequently also for the velocity field.

To prove this, we will use the variational formulation of the problem for the stream function, which it is easy to obtain from general principles for viscoplastic media [7, 8].

The stream function ψ, which satisfies all the conditions and Eqs (2.1)–(2.4), gives an absolute minimum of the functional I

$$I=\int_0^l dx\int_0^{2h} Ddz,\quad D=\frac{1}{2}\mu(\psi'')^2+\tau_0|\psi''| \tag{2.5}$$

among all the doubly differentiable functions ($\psi(x, z) \in C_2$) which satisfy the boundary conditions

$$\psi(0,z)=\psi(x,0)=\psi'(x,0)=\psi'(x,2h)=0,\quad \psi(x,2h)=-2\dot{h}x=Q(x) \tag{2.6}$$

where $Q(x)$ is the flow rate across a section with coordinate x. A partial derivative with respect to z is denoted by a prime.

The dissipative potential possesses the following properties: strict convexity and boundedness from below ($D \geqslant 0$). The strict convexity, the boundedness from below of the functional I and the existence of a unique minimizing element for it hence follows from this. The derivative of D with respect to ψ'' corresponds to the shear stress τ_{xz} determined using (2.2), that is

$$\partial D/\partial\psi''=\tau_{xz}(\psi'') \tag{2.7}$$

(The concept of subdifferentiability, which generalizes conventional differentiation [9, pp. 30–37], has to be used to determine the non-smooth function in (2.7).)

Using property (2.7), it can be shown that the minimizing element of the variational problem (2.5), (2.6) satisfies all the equations and conditions of (2.1)–(2.4).

Actually, for a variation in D we have the identity

$$\delta D=\frac{\partial D}{\partial\psi}\delta\psi''=\tau_{xz}(\psi'')\delta\psi''=\frac{\partial}{\partial z}\left(\tau_{xz}\delta\psi'-\frac{\partial\tau_{xz}}{\partial z}\delta\psi\right)+\frac{\partial^2\tau_{xz}}{\partial z^2}\delta\psi \tag{2.8}$$

By virtue of (2.6), the variations of ψ and its derivative on the boundaries $z = 0$ and $z = 2h$ are equal to zero: $\delta\psi = \delta\psi' = 0$. From (2.8), for a variation in the functional (2.5), we therefore obtain

$$\delta I=\int_0^l dx\int_0^{2h}\frac{\partial^2\tau_{xz}}{\partial z^2}\delta\psi dz$$

From the necessary condition for an extremum $\delta I = 0$, we obtain the equation for the minimizing element

$$\partial^2\tau_{xz}/\partial z^2=0 \tag{2.9}$$

It is clear that Eq. (2.9) and the function $\tau_{xz}(\psi'')$, determined using (2.2), and the boundary conditions (2.6) are equivalent to the equations and boundary conditions for ψ following from (2.1)–(2.4).

Hence, the equivalence of problem (2.1)–(2.4) in the case of the stream function and the variational problem (2.5)–(2.6) is proved and the theorem on the existence and uniqueness of the stream function ψ is thereby also proved.

3. SOLUTION OF THE BOUNDARY-VALUE PROBLEM

If, in the last equation of (2.6), the function $Q(x)$ is replaced by the constant quantity Q, then Eqs (2.9) and (2.2), with conditions (2.6), represent the exact boundary-value problem for the stream function of the steady-state flow of a viscoplastic medium between two parallel fixed plates. Its exact solution is given by formulae (1.2)–(1.4), which can be shown to be so by checking. Note that the variable x occurs in problem (2.9), (2.2), (2.6) as a parameter and its solution for the stream function is therefore represented by formulae (1.2) and (1.4), where Q and a are functions of the x coordinate which correspond to the last equation of (2.6), that is

$$Q=-2\dot{h}x,\quad a=\left|\frac{2\mu\dot{h}}{\tau_0 h^2}x\right| \tag{3.1}$$

In the case of the function $Z_0(a)$, all of the results obtained in Section 1 are preserved and, in fact: $Z_0(a) \in [0, 1)$ is the single root of the cubic equation (1.5) with the parameters a; the functions $Z_0(a)$ and $(1 - Z_0(a))^{-1}$ are defined by series (1.6), which converge for all values of the argument a, or by the asymptotic expansions (1.7) when $a \leqslant 1$.

A solution of the problem of the flow of a viscoplastic medium between two plates as they come closer together or move farther apart has therefore been obtained. The flow in an arbitrary cross-section is completely defined by a single dimensionless parameter a, which has the meaning of the local inverse Saint-Venant number. In the problem under consideration, there is a single similarity parameter a [10].

As the two plates approach one another, the flow in a section with a similarity parameter a will similarly be a flow between two fixed parallel plates with a flow rate which corresponds to the same value of a (Fig. 1a, b). The velocity components are determined from (1.2) and (1.3)

$$v_x = -\frac{h\tau_0 \operatorname{sign}\dot{h}}{6\mu}\frac{\partial\Psi}{\partial Z}, \quad v_z = -\frac{\partial\psi}{\partial x} = \frac{\dot{h}}{3}\frac{\partial\Psi}{\partial Z_0}\frac{dZ_0}{da} \tag{3.2}$$

The derivative dZ_0/da in (3.2) is found by differentiating the first series of (1.6) or the first expansion of (1.7).

Error estimates. The thin-layer approximation assumes that the inertial force $\rho(\partial v_x/\partial t + v_x\partial v_x/\partial x) \sim \rho v_x^2/l$ is small compared with the force due to viscous drag $\partial\tau_{xz}/\partial z \sim \mu v_x/(h^2Z_0^2)$. The velocity is estimated by the quantity $v_x \sim \dot{h}l/h$. In its order of magnitude, the error which occurs when the inertial forces are neglected is equal to the ratio of the inertial forces and the force due to viscous drag, that is, to the Reynolds number

$$\mathrm{Re} = \rho v_x h^2 Z_0^2/(\mu l) = \rho\dot{h}hZ_0^2/\mu$$

Hence, the result which has been obtained will be approximate for the condition $\mathrm{Re} \ll 1$ and $h/l \ll 1$. In the case of the purely plastic solution, ($Z_0 = 0$), the first condition is always satisfied and the condition $h/l \ll 1$ alone is sufficient.

Analysis of the solution. Exact expressions for the stresses in the problem of the extrusion of a purely plastic medium have been obtained by Prandtl [2]

$$p = p_0 - \tau_0(x/h + 2\sqrt{1-(z-h)^2/h^2}), \qquad \tau_{xz} = \tau_0(z-h)/h$$

We shall now present a comparison of the asymptotic solution of (1.2)–(1.7), (3.1) and known solutions in different limiting cases.

In the case of purely plastic flow, the pressure is found from the solution of (1.3) which is obtained when $Z_0 = 0$ and is equal to $p = p_0 - \tau_0 x/h$. It differs from the exact pressure by a small quantity of the order of h/l while the shear stress found using (1.3) is identical with the Prandtl solution [2].

Taking account of viscous forces in the case of a large Saint-Venant number [7] corresponds to the approximation $a \ll 1$. It follows from (1.4) that $Z_0 \approx \sqrt{a}$. On substituting this asymptotic form into (1.2) and (1.3), we obtain a two-term expansion, which corresponds to the solution obtained previously [7].

In the case of purely viscous flow, the solution corresponds exactly to the solution of (1.2), (1.3), (3.1) if the limiting value $Z_0 = 1$ is substituted into it.

The solution which has been presented for the stresses (1.4)–(1.7), (3.1) and for the velocities (1.2), (1.3) holds over the whole range of Saint-Venant numbers and includes all solutions which are known up to this time if the distance between the plates is small compared with their size.

4. THE FORCE

The force which acts per unit width of a plate can be calculated by integrating the pressure on its surface, partially transforming the corresponding integral and then using the first relation of (1.4)

$$F = 2\int_0^l (p - p_l)dx = -2\int_0^l x\frac{\partial p}{\partial x}dx = -\operatorname{sign}\dot{h}\frac{\tau_0 l^2}{h}\frac{2f(a_l)}{a_l^2} = F_0\frac{2f(a_l)}{a_l^3} = F_\infty\frac{2f(a_l)}{a_l^2} \tag{4.1}$$

$$F_0 = -2\left(\frac{l}{h}\right)^3\mu\dot{h}, \quad F_\infty = -\frac{\tau_0 l^2}{h}\operatorname{sign}\dot{h}, \quad f(a) = \int_0^a \frac{a\,da}{1 - Z_0(a)}, \quad a_l = \frac{2\mu|\dot{h}|}{\tau_0 h^2}l \tag{4.2}$$

where p_l is the pressure on the boundary of a plate when $x = l$, a_l is the inverse Saint-Venant number, and F_0 and F_∞ are the forces for the cases of a purely viscous medium ($\tau_0 = 0$, $a_l = \infty$) and a purely plastic medium ($\mu = 0$, $a_l = 0$), respectively.

On substituting the second series of (1.6) or the second expansion of (1.7) into integral (4.2), we find

$$f(a)=\frac{1}{2}a^3+\frac{3}{4}a^2-\frac{2}{9}\ln(a+1)+\alpha+\sum_{m=1}^{\infty}\frac{b_m}{3m(a+1)^{3m}}-\sum_{m=0}^{\infty}\frac{b_m}{(3m+1)(a+1)^{3m+1}} \tag{4.3}$$

$$\alpha=\frac{2}{9}-\sum_{m=1}^{\infty}\frac{b_m}{3m(3m+1)}=0.21546$$

$$f(a)=\frac{1}{2}a^2+\frac{2}{5}a^{5/2}+\frac{2}{9}a^3+\frac{5}{63}a^{7/2}+\frac{1}{4\cdot 27}a^4+\ldots,\quad a<1 \tag{4.4}$$

Series (4.3) converges for all $a \geqslant 0$. It is more convenient to make use of expansion (4.4) when $a < 1$. It follows from the result obtained that the force is determined by the single similarity parameter a_l.

For high Saint-Venant numbers ($a_l \ll 1$) and the (4.1), (4.2) and (4.4) and (4.6) for them, we obtain the asymptotic expansion found in [3]

$$F=F_\infty\left(1+\frac{4}{5}\sqrt{\frac{2l|\mu|\dot{h}|}{\tau_0 h^2}}\right)$$

5. MOTION OF THE PLATES UNDER A CONSTANT FORCE

In the case of a specified constant force F, expressions (4.1) and (4.2) change into a differential equation in $h(t)$. Actually, expression (4.1) can be reduced to the form

$$\frac{h}{h_0}=\frac{2f(a_l)}{a_l^2},\quad h_0=\frac{\tau_0 l^2}{|F|} \tag{5.1}$$

By virtue of the inequality $2f(a_l)/a_l^2 > 1$, we conclude that $h > h_0$, that is, h_0 is the shortest distance the plates can approach under the force F. Starting out from its definition (4.2), the variable a_l can be represented in terms of a time derivative as follows:

$$a_l=\left|\frac{d}{dt}\frac{h_0 t_0}{h}\right|,\quad h_0 t_0=\frac{\mu l}{\tau_0},\quad t_0=\frac{\mu|F|}{\tau_0^2 l} \tag{5.2}$$

Then, substituting (5.2) into relation (5.1), we obtain a differential equation for the dimensionless function h_0/h of the dimensionless time t/t_0. It can be shown by checking that the solution of this equation can be represented in the following parametric form

$$\frac{h_0}{h}=\Phi(a_l),\quad \frac{t}{t_0}=\operatorname{sign}\dot{h}\int_{a_l}^{\infty}\frac{\Phi'(a)da}{a},\quad \left(\Phi(a)=\frac{a^2}{2f(a)}\right) \tag{5.3}$$

Asymptotic expansions can be obtained for sufficiently small a

$$\Phi(a)=1-\frac{4}{5}a^{1/2}+\frac{44}{225}a+\sum_{n=1}^{\infty}\Phi_n a^{1+n/2} \tag{5.4}$$

$$\frac{t}{t_0}=\frac{4}{5}a^{-1/2}+\frac{44}{225}\ln a+b+\sum_{n=1}^{\infty}\left(\frac{2}{n}+1\right)\Phi_n a^{n/2}$$

$$b=-\frac{91}{225}-\int_0^1\left(\sum_{n=1}^{\infty}\Phi_n a^{-1+n/2}\right)da-\int_1^{\infty}\frac{\Phi(a)}{a^2}da=-0.7328$$

The values of the first seven coefficients Φ_n in expansion (5.4) are 0.0404; –0.0107; –0.0012; –0.0013; 0.0027; 0.0014; –0.0003.

When $a < 1.4$, the function $t(a)$ can be calculated with an error of less than 1% using formula (5.4). When $a \geqslant 2$, calculations with the same error can be carried out using the following asymptotic formula

$$t/t_0=a/(2f(a))+\tfrac{4}{9}\ln(1+3/(2a))-2/(3a)$$

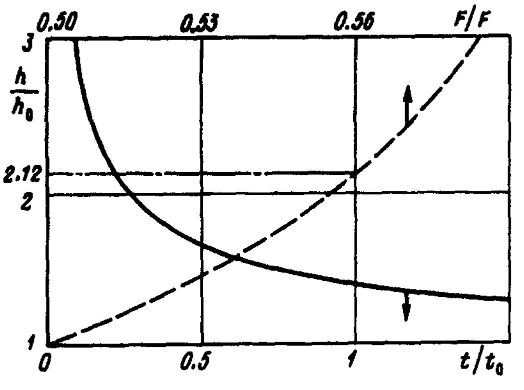

Fig. 2.

It has therefore been established that the motion of the plates for any values of the force, coefficient of viscosity, yield stress and geometrical characteristics is defined by the single, universal dependence of h/h_0 on t/t_0. The form of this dependence is represented by the solid line in Fig. 2.

6. THE CONDITION FOR CONTINUOUS SEPARATION OF THE PLATES

Separation of the plates under a constant force occurs along the same trajectory $h(t)$ as when they approach each other but in the opposite direction. However, by virtue of the drop in the pressure on the surface of the plates to zero, cavities (flow separation) may be formed in certain segments of the trajectory. Using (1.4) and (2.4), the condition for continuous motion can be written as

$$p(0) = p_l - \int_0^l \frac{\tau_0}{h}\frac{dx}{1-Z_0} > 0 \Rightarrow h \geqslant \frac{\tau_0 l}{p_l}\frac{P(a_l)}{a_l}, \quad P(a_l) = \int_0^{a_l}\frac{da}{1-Z_0(a)} \tag{6.1}$$

(the substitution $dx = l da/a_l$ has been used here).

On substituting the law of motion (5.1) instead of h into the inequality, we obtain the condition for continuous separation of the plates

$$G(a_l) = \frac{a_l}{f(a_l)}P(a_l) \leqslant \frac{F_l}{F}, \quad F_l = 2lp_l \tag{6.2}$$

Substituting the second series of (1.6) or the second expansion of (1.7) into integral (6.1), we obtain a series and an expansions for the integral in (6.1)

$$P(a) = \frac{3}{4}a^2 + \frac{3}{2}a - \beta + \sum_{m=0}^{\infty}\frac{b_m}{(3m+1)(a+1)^{3m+1}} \tag{6.3}$$

$$\beta = \sum_{m=0}^{\infty}\frac{b_m}{3m+1} = 0.2496$$

$$P(a) = a + \frac{2}{3}a^{3/2} + \frac{1}{3}a^2 + \frac{1}{9}a^{5/2} + \frac{1}{81}a^3 + \ldots \quad (a<1) \tag{6.4}$$

Expansions (6.3) and (6.4), and expansions (4.3) and (4.4) for $f(a)$ provide comprehensive information on the function $G(a)$. When the argument $0 \leqslant a < \infty$ is changed, the function $G(a)$ decreases monotonically from the value $G(0) = 2$ to the value $G(\infty) = 3/2$. It follows from this that, when $F \leqslant F_l/2$, the separation of the plates will always be continuous. On the other hand, when $F_l/2 \leqslant F \leqslant 2F_l/3$, the separation of the plates will be continuous when $h \geqslant h_*$. The critical value h_* is determined from the simultaneous solution of Eq. (5.1) and an equation which follows from (6.1), that is

$$\frac{h_*}{h_0} = \frac{2f(a_l)}{a_l^2}, \quad \frac{f(a_l)}{a_l P(a_l)} = \frac{F}{F_l}$$

The dependence of h/h_0 on F/F_l is represented by the dashed line in Fig. 2. The scale of the variable $1/2 \leqslant F/F_l \leqslant 2/3$ is shown along the top of this figure. (For example, when $F/F_l = 0.56$, the flow will be continuous along the part of the trajectory $h/h_0 > h/h_* \approx 2.12$.)

7. THE ROOTS OF EQ. (1.5)

The function $a(Z_0)$, specified by formula (1.5), increases monotonically in the interval [0, 1) and takes any non-negative value just once. An inverse function $Z_0(a) \in [0, 1)$ exists for it, and this determines the single root of Eq. (1.5) from the interval [0, 1) for any $a \geqslant 0$.

To construct series (1.6), which correspond to the root of Eq. (1.5) from the interval $Z_0(a) \in [0, 1)$, we transform Eq. (1.5) to the form

$$3(a+1)=\frac{(1-Z_0)^3+2}{1-Z_0} \tag{7.1}$$

and make use of Lagrange's theorem [11, pp. 507–511] on the inversion of series.

Suppose there is an equation in y of the form

$$y=A+x\Phi(y) \tag{7.2}$$

where x is a variable, A is a constant and $\Phi(y)$ is a function which is analytic at the point $y = A$. Then, a neighbourhood $x \in (-\varepsilon, \varepsilon)$ exists in which the root of Eq. (7.2) is represented by the series

$$e=A+x\Phi(A)+\sum_{n=2}^{\infty}\frac{x^n}{n!}\frac{d^{n-1}}{dA^{n-1}}[\Phi^n(A)] \tag{7.3}$$

To obtain series (1.6), it is sufficient to transform Eq. (7.1) to the form (7.2) and then calculate the coefficients of the Lagrange series (7.3).

Using the substitution

$$x=\frac{1}{3(a+1)}, \quad y=1-Z_0, \quad \Phi(y)=y^3+2, \quad A=0 \tag{7.4}$$

Equation (7.1) is reduced to the form (7.2).

The derivatives of $\Phi^n(A)$ in (7.2) at the point $A = 0$ are expressed in terms of the binomial coefficients C^m_{3m+1}. On substituting the expressions for them into the Lagrange series (7.3), we obtain the first series of (1.6).

A second change of the variables in Eq. (7.1)

$$x=\frac{-4}{27(a+1)^3}, \quad y=\frac{2}{3(a+1)(1-Z_0)}, \quad \Phi(y)=y^{-2}, \quad A=1 \tag{7.5}$$

also reduces it to the form (7.2). Substituting (7.5) into the Lagrange series (7.2) and calculating the derivatives of the function $\Phi^n(A) = A^{-2n}$ at the point $A = 1$, we obtain the second series of (1.6).

We will now show how to construct the best approximations using partial sums of series (1.6) and expansions (1.7) taking the example of four-term expansions

$$Z_0=1-\frac{2}{3(a+1)}-\frac{8}{81(a+1)^4}-\frac{32}{3^6(a+1)^7}, \quad a>a_0 \tag{7.6}$$

$$Z_0=a^{1/2}-\frac{1}{3}a-\frac{1}{18}a^{3/2}+\frac{1}{27}a^2, \quad a<a_0$$

The problem consists of determining the boundary point a_0, at which formulae (7.6) determine the function $Z_0(a)$ with the least error. The error of the approximation (7.6) when $a > a_0$ and $a < a_0$ is estimated by the quantities

Table 1

	Z_0		$(1-Z_0)^{-1}$	
n	a_0	$\lvert r_+\rvert = \lvert r_-\rvert$	a_0	$\lvert r_+\rvert = \lvert r_-\rvert$
2	0.50	0.0200	0.070	0.0460
3	0.37	0.0050	0.130	0.0130
4	0.33	0.0015	0.237	0.0021

$$r_+ \approx \frac{-2^9}{3^9(a+1)^{10}}, \quad r_- \approx \frac{5}{216a^{5/2}}$$

The best approximation and the greatest error are found from the equation $|r_+(a)| = |r_-(a)|$. From this, we obtain that $a_0 = 0.33$ and that the greatest error is $|r_+| = |r_-| = 0.0015$.

The best three-term approximation can be constructed in an analogous manner. For this approximation, we obtain the greatest error $|r_+| = |r_-| = 0.005$ when $a_0 = 0.366$.

In exactly the same way, we can find the best two-, three- and four-term approximations for the function $1/(1 - Z_0)$. The results are shown in Table 1. The form of the functions is shown in the top line. The number of terms in the partial sums of series (1.6) and expansions (1.7) is indicated in the first column. The values of a_0 and the greatest error $r_\pm$ are indicated in the following columns. Taking account of each successive term reduces the error by a factor of approximately four.

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